Announcements

1) Math Career Talks
   Today
   2:30 CB 1030

2) Supplement to HW 3
   up later today.

Recall: Two vectors v and w in $\mathbb{R}^n$ are orthogonal if

$$v \cdot w = 0$$

Orthogonality and Linear Independence

If $\{v_1, v_2, \ldots, v_m\}$ are orthogonal in $\mathbb{R}^n$, then $\{v_1, v_2, \ldots, v_m\}$ is linearly independent provided none of the vectors are the zero vector.

Why is this true?

Remember we know that $\{v_1, v_2, \dots, v_m\}$ is linearly independent whenever

$$a_1 v_1 + a_2 v + \cdots + a_m v_m = \vec{0}$$

implies $a_1 = a_2 = \cdots = a_m = 0$.

Let's suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = \vec{0}.$$

Pick an i, $1 \le i \le m$.

Then dot product both sides of

$$a_1 v_1 + a_2 v_2 + \cdots + a_m v_m = \vec{0}.$$

We get

$$v_i \cdot (a_1 v_1 + a_2 v_2 + \cdots + a_m v_m)$$

$$= v_i \cdot \vec{0} = 0$$

But $v_i \cdot (a_1 v_1 + a_2 v_2 + \cdots + a_m v_n)$

$$= a_1 (v_i \cdot v_1) + a_2 (v_i \cdot v_2) + \cdots + a_m (v_i \cdot v_m)$$

$= a_i \, v_i \cdot v_i$ since $v_i \cdot v_j = 0$ if $i \neq j$.

This says

$$a_i (v_i \cdot v_i) = 0.$$

But if $v_i \neq 0$,

$$v_i \cdot v_i = \| v_i \|_2^2 > 0.$$

Since $v_i \cdot v_i \neq 0$ and $a_i (v_i \cdot v_i) = 0$, we have $a_i = 0$ for all $1 \leq i \leq m$, so we have linear independence.

Example 1:

Given $v = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ in $\mathbb{R}^2$,

find a nonzero vector orthogonal to v.

Flip coordinates, put a negative sign on one of them

$$w = \begin{bmatrix} 6 \\ -7 \end{bmatrix}.$$

Then

$$v \cdot w = 7 \cdot 6 + 6 \cdot (-7)$$
$$= 0$$

In fact, every vector in $\mathbb{R}^2$ orthogonal to v is a scalar multiple of w.

Example 2: If $v = \begin{bmatrix} 3 \\ 4 \\ 15 \\ 56 \end{bmatrix}$ is in $\mathbb{R}^4$, find a nonzero vector w orthogonal to v.

$$w = \begin{bmatrix} 4 \\ -3 \\ 56 \\ -15 \end{bmatrix}$$

$v \cdot w = 0$, so done.

$$\begin{bmatrix} 3 \\ 4 \\ 15 \\ 56 \end{bmatrix}$$

$$w_1 = \begin{bmatrix} 56 \\ 15 \\ -4 \\ -3 \end{bmatrix}$$

is another vector orthogonal to v that is not a scalar multiple of w.

Example 3: $v = \begin{bmatrix} 52 \\ 74 \\ -3 \end{bmatrix}$

in $\mathbb{R}^3$. Find a nonzero vector orthogonal to v.

$$w = \begin{bmatrix} -74 \\ 52 \\ 0 \end{bmatrix}$$ works

Definition: (transpose)

Let A be an $m \times n$ matrix. The transpose of A, written A^t, is the $n \times m$ matrix whose rows are the columns of A.

Example 4:

Let $A = \begin{bmatrix} -2 & 1 \\ 3 & 8 \\ 6 & 0 \end{bmatrix}$

(3×2)

$$A^t = \begin{bmatrix} -2 & 3 & 6 \\ 1 & 8 & 0 \end{bmatrix}$$

(2×3)

$$A = \begin{bmatrix} -3 & 16 & 2 \\ 5 & 8 & 4 \\ 1 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

(4×3)

$$A^t = \begin{bmatrix} -3 & 5 & 1 & 0 \\ 16 & 8 & 3 & 0 \\ 2 & 4 & 6 & 5 \end{bmatrix}$$

(3×4)

$$A = \begin{bmatrix} 1 & 15 & 4 \\ 15 & 8 & -100 \\ 4 & -100 & 6 \end{bmatrix}$$

(3×3)

$$A^t = \begin{bmatrix} 1 & 15 & 4 \\ 15 & 8 & -100 \\ 4 & -100 & 6 \end{bmatrix}$$

$$= A \,!$$

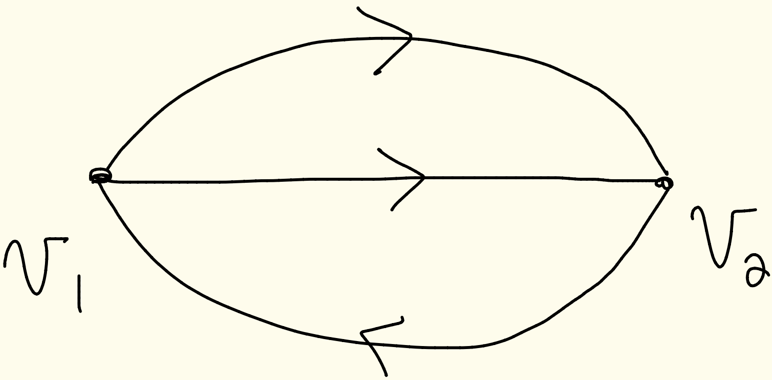
This is a special kind of matrix.

Electrical Circuits

Our circuits will have no capacitors, only resistors, batteries, and current sources.

We start with a collection of nodes and edges that connect them

Example 5:



2 nodes, 3 edges.

Arrow indicates direction.

Edge-Node Incidence Matrix

m rows, one for every edge.

n columns, one for every node

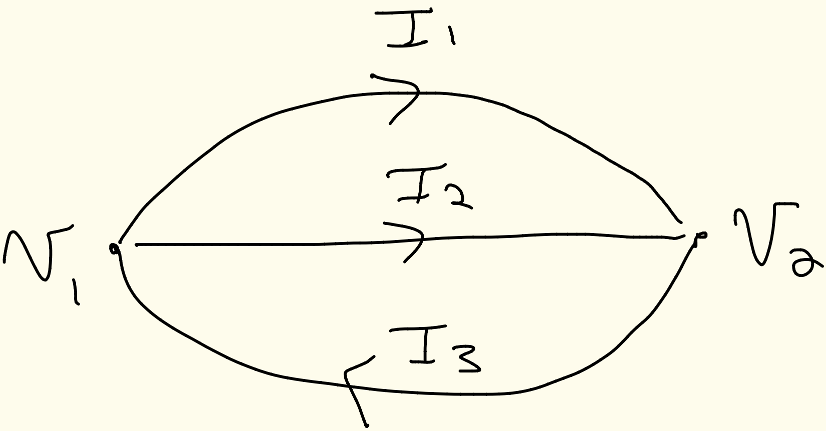
If $1 \leq i \leq m$ and $1 \leq j \leq n$, the (i, j) (i^{th} row, j^{th} column) of the matrix will be

$$\begin{cases} 1, & \text{if edge } i \text{ ends in node } j \\ -1, & \text{if edge } i \text{ begins in node } j \\ 0, & \text{if edge } i \text{ is not connected to node } j \end{cases}$$

Call this entry $A_{i,j}$ and the resulting matrix A.

$$A = (A_{i,j})_{i=1, j=1}^{m \quad n}$$

Back to Example 5



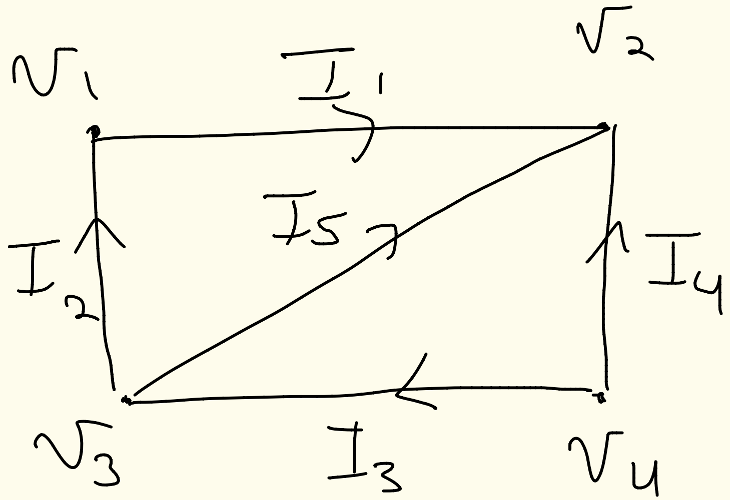
nodes v_1, v_2

edges I_1, I_2, and I_3

$$A = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \\ I_3 \end{array} & \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}$$

(3×2)

Example 6:



$$A = \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$